on the deformation of the cylinder. As the ratio R'/R is fairly large in the actual cases considered, the main term expressing the cylinder distortion is proportional to $(1+\sigma)/E$, i. e. to $1/G$ where G is the modulus of rigidity.

Interpreting k as the quotient of the two moduli of rigidity, we now have

$$\begin{aligned}\lambda_A - \lambda_B &= \varphi_A - \varphi_B + \theta_A - \theta_B \\ &= (1-\mathrm{k})\,\varphi_A + \theta_A - \theta_B \\ &= (1-\mathrm{k})\,\lambda_A + (k\theta_A - \theta_B) \qquad (3.5)\end{aligned}$$

determining λ_A in terms of the difference coefficient $\lambda_A - \lambda_B$ established by the balancing procedure, the value of k, and the correction term $(k\theta_A - \theta_B)$.

d) Extension to the use of three materials

In the first series of experiments the material adopted for the comparison assemblies was a form of aluminium bronze, known commercially as "hydurax", the modulus of rigidity of which was lower than that of steel in the ratio 1:1.44. The Poisson's ratio was rather higher than that of steel (see Tab. 1 for further details). It was apparent that a check involving a third material, differing substantially in elastic properties from those used hitherto, would provide a valuable test of the accuracy of the similarity method. An even better check would naturally be provided by a completely independent pair of materials. This latter extension has not so far been found practicable as the choice of materials possessing all the qualities required is limited. It has been found possible, however, to extend the procedure to include three materials, the third being an alloy of tungsten known commercially as "GEC Heavy Metal". This material proved to have a high degree of isotropy and a Poisson's ratio very close to that of the material used for the steel assemblies. The elastic moduli exceed those of steel in about the ratio 1.75:1 and it was of advantage that in this case the comparison should involve a material having a modulus higher than that of steel in contrast to the former comparisons in which the reverse held.

In discussing this extension of the method it will be convenient to refer to the steel, bronze and tungsten assemblies by the initial letters S, B and T respectively. With a group of three materials, the distortion coefficient of any one assembly, say S, may be reached by three different routes, two of them direct – i. e. involving direct comparisons with the other two assemblies B and T – and the other indirect. In the latter procedure the distortion coefficient of one of the other two assemblies, say T, is first determined by applying the similarity principle to B and T, and the coefficient for S is then obtained by simple addition of the difference coefficient for S and T. It is of interest to note that the indirect procedure leads to the distortion coefficient of the assembly chosen, i. e. S, without any appeal to the elastic constants of the material of S. These three derivations are not entirely independent but, since the six independent elastic moduli involve five independent ratios, no one result is in general deducible from the other two. Proceeding on the lines of equation (3.5) and denoting by $\lambda_S \ldots$ the true values of the distortion coefficients, $G_S \ldots$ the moduli of rigidity, $\theta_S \ldots$ the corresponding correction terms given by equation (3.4), λ_{SB} ($= \lambda_S - \lambda_B$) ... the difference coefficients determined by the balancing experiments, and $k_{SB} = G_B/G_S \ldots$, we have for the three possible experimental values, λ'_S, λ''_S and λ'''_S, of λ_S, the equations

$$\lambda'_S\,(k_{BS} - 1) = \lambda_{BS} - \theta_B + k_{BS}\,\theta_S \qquad (3.6)$$

$$\lambda''_S\,(k_{TS} - 1) = \lambda_{TS} - \theta_T + k_{TS}\,\theta_S \qquad (3.7)$$

for the direct comparisons, and for the indirect

Table 1. *Summary of elastic constants*

Material	Modulus of rigidity (G) (dyn/cm²)*		Young's modulus (E) (dyn/cm²)*		Poisson's ratio (σ) (Ultrasonic pulse method)
	Torsion extensometer method	Ultrasonic pulse method	Extensometer method (mean of results for tension and compression)	Ultrasonic pulse method	
Steel (K 9) (hardened and tempered)	7.86×10^{11}	7.92×10^{11}	20.6×10^{11}	20.5×10^{11}	0.295
Aluminium bronze ("hydurax")	5.45×10^{11}	5.38×10^{11}	$14.4_5 \times 10^{11}$	$14.3_3 \times 10^{11}$	0.333
Tungsten alloy ("GEC heavy metal" – specific gravity 18)	$13.5_5 \times 10^{11}$	$14.2_5 \times 10^{11}$	36.1×10^{11}	36.7×10^{11}	0.286_5

* 1 dyn/cm² = 0.1 N/m².

$$\lambda'''_S\,(k_{BT} - 1) = \lambda_{BT} - \theta_B + k_{BT}\,\theta_T + \lambda_{ST}\,(k_{BT} - 1). \qquad (3.8)$$

Transposing these equations and making use of the subsidiary relations

$$\left.\begin{aligned} k_{BS}\,k_{SB} = 1 \ldots\;; \quad k_{BS}\,k_{ST}\,k_{TB} = 1\;; \\ \lambda_{BS} = -\lambda_{SB} \ldots\;; \quad \lambda_{BS} + \lambda_{ST} + \lambda_{TB} = 0\;; \end{aligned}\right\} \qquad (3.9)$$

we eventually obtain

$$(\lambda'_S - \lambda'''_S)\,(1 - k_{SB}) = (\lambda'''_S - \lambda''_S)\,(k_{ST} - 1). \qquad (3.10)$$

Since $(1 - k_{SB})$ and $(k_{ST} - 1)$ are both positive it easily follows from this equation that the three values λ'_S, λ''_S and λ'''_S must either be all equal or all unequal, and that the indirect value λ'''_S must be intermediate between the two direct values, whatever the nature of the experimental errors*. The practical significance of various possible errors is examined in more detail in section 4 b).

e) Determination of elastic constants

The elastic constants utilised in the investigation were measured in the Strength of Materials Section of the Basic

* We ignore cases where either k_{SB} or k_{ST} is so close to unity that experimental errors might cause a change of sign of (1-k_{SB}) or (k_{ST}-1) since such conditions would not be acceptable as a basis for the similarity method.

Physics Division of the National Physical Laboratory, and included results obtained by the ultrasonic pulse method (MARKHAM 1957) as well as by the standard static methods giving the stress-strain relations over a wide range of stress. Young's modulus was measured both in tension and compression using a Martens type rhomb and mirror extensometer. The modulus of rigidity was determined by means of an NPL design of torsion extensometer in which readings were taken either with an autocollimator or with the normal arrangement of scale and telescopes.

Precautions were taken to ensure that the samples used for the preparation of test pieces were sufficiently representative of the material used in the piston-cylinder assemblies. Wherever possible they were selected from the same piece or batch of material. In cases where this was impracticable, material of similar composition was used, care being taken that any heat treatments involved were adequately reproduced. A study by BROWN, COLE & MARKHAM (1957) on the effects of heat treatment and tempering on the elastic moduli of the steels concerned illustrates the significance of these effects.

The results of the elastic modulus measurements are summarised in Tab. 1. On the whole the agreement between the ultrasonic and static methods is good, the discrepancies rarely exceeding 1 or 2%. It seemed desirable, however, to decide on a consistent basis for the choice of the actual values to be adopted in practice, especially as regards the values of G and σ which are particularly important in the applications to the similarity method. It was decided, after consultation with experts in the field of elastic properties, to proceed as follows:

i) For the modulus of rigidity, to adopt the static values taken over a wide range of stress, as being those most likely to be representative of the conditions obtaining in practice when the system is subjected to sustained forces. It is pertinent to note that as we are interested only in the *ratio* of the values of G for a pair of materials, certain types of systematic error in the elastic measurements will be eliminated.

ii) For Poisson's ratio, to adopt the values obtained by the ultrasonic method in which this quantity is given directly in terms of the observed wave velocities. This value is likely to be considerably more accurate than one derived indirectly from static measurements of E and G since, as these are determined by different experimental procedures, their ratio may be subject to a systematic error. Since $E/G = 2(1 + \sigma)$ and σ is normally intermediate between 1/3 and 1/4, any error in E/G would entail an error proportionately 4 or 5 times larger in σ. It may be noted, however, that even if the actual value of E/G were somewhat in error the relation between the loads and displacements would still help to show up any important *variation* in σ over the range of stress, so that the static results provide useful evidence on this point.

The ultrasonic measurements provide direct information on the elatic isotropy of the material. This was found to be satisfactory in the case of all three materials considered in this investigation.

The relations between displacement and applied force given by the extensometer measurements showed a satisfactory degree of linearity, and freedom from important hysteresis effects, with the exception of the tungsten alloy at high stresses. When tested under the condition of a rising series of values of stress, this material exhibited departures from linearity, principally for stresses above about 1600 bars (1.6×10^8 N/m^2), which seemed consistent with some degree of plastic deformation. Series taken in descending order of stress, however, showed a much closer approximation to linear behaviour, indicating a modulus reasonably consistent with that obtaining over the lower range of stress, i. e. before the appearance of the anomalous permanent set. This point is further discussed in the next section, where a variation of the balancing procedure used in the similarity method, to take account of this anomaly, is described.

f) Experimental method

As previously remarked, the effective areas of the piston-cylinder assemblies of the two different materials have been compared by direct balancing on a common pressure system as this is the most convenient method assuming that two complete pressure balances are available*.

For the purposes of the present work the equilibrium state of a piston-cylinder assembly is defined to be that in which the piston is falling at such a rate as exactly to compensate for the volume of fluid lost by the natural leakage through the interspace between the piston and cylinder. In the case of two assemblies balanced against one another, these conditions imply that there is no movement of fluid through the connecting line. Leaks in other parts of the system must of course be carefully controlled if these equilibrium conditions are to be reproduced unambiguously. The accuracy of the balancing process is normally of the order of a few parts in 10^6.

The dependence of the effective area on temperature has been found to be adequately represented by the area coefficient of thermal dilatation which, in the case of steel assemblies, amounts to a change of about 2.3 parts in 10^5/ °C. The temperatures of the piston-cylinder assemblies were measured to within about 0.05 °C.

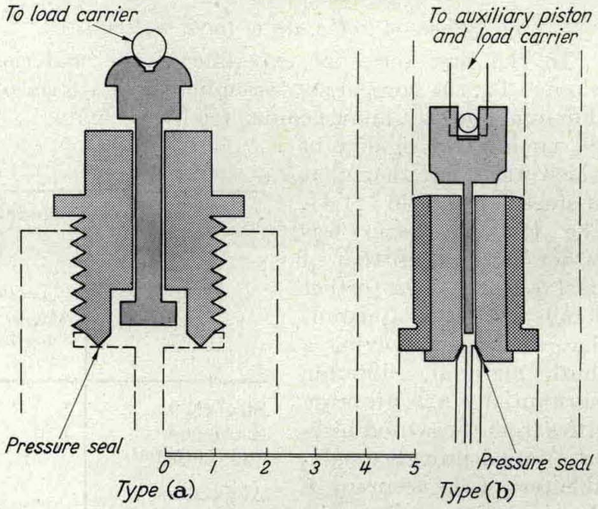

Fig. 2. Diagrams of piston-cylinder assemblies (Scale of cm)

Some obvious small corrections to the loads on the two assemblies may be necessary to account for:

i) any difference of level of the two pistons;

ii) buoyancy effects due to any submerged portions of the piston of other than the working diameter;

iii) surface tension at the meniscus at the upper end of the piston.

Since the comparison is between assemblies of the same nominal dimensions, the corrections involved in ii) and iii) will normally cancel out, or nearly so.

Two rather different types of piston-cylinder assembly have been used in the present work, and these are shown diagrammatically in Fig. 2, a) and b). Units of type a) have been used over the range of pressure up to about 3000 bars, the assemblies having nominal effective areas of 0.05, 0.02 and 0.01 in^2* and differing only in the diameter of the piston and cylinder bore. The units of type b), which have been used mainly for the higher part of the pressure range – i. e. from about 1500 to 6000 bars – were of nominal area 0.005 in^2*.

The piston-cylinder units of type a) are attached to the support column by screwing into a collar shown in outline in Fig. 2, the pressure seal being effected between an annular projection at the base of the assembly and a flat shelf at the upper end of the column. In order to avoid any possibility of anomalous effects due to a discontinuity in the elastic modulus at the junction, the support column used in association with any particular assembly was constructed of the same material as the assembly itself. In the units of type b) the housing, also shown in Fig. 2, was rather different. The main cylinder block

* It should be noted that the balancing process is not in itself fundamental to the similarity procedure. The essential condition is that the equilibrating loads on the two assemblies should be determined for *exactly* the same pressure. It would be possible, though more difficult, to do this by determining the load on each assembly separately when exposed to an accurately reproducible pressure identified, for example, by a phase transition of a pure substance. If two complete balances were not available it might well be necessary to resort to some such method.

* The approximate metric equivalents are: 0.05 in^2 = 0.322 cm^2; 0.02 in^2 = 0.129 cm^2; 0.01 in^2 = 0.0645 cm^2; 0,005 in^2 = 0.0322 cm^2.